Classical and Quantum Chiral Order in Frustrated XY Magnets

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Abstract

Recent studies on the chiral order of regularly frustrated XY magnets are reviewed both in classical and quantum cases. In the classical case, chiral transition is a thermal one, while in the quantum case, it is a quantum phase transition. Importance of spatial dimensionality on the chiral order is clarified. Particular attention is paid to the possible "spin-chirality decoupling" phenomenon, and to the possible pure chiral phase of either thermal or quantum origin where the chirality exhibits a long-range order without the standard spin order.

1. Introduction

Magnetic ordering of geometrically frustrated antiferromagnets has attracted continual interest of researchers in magnetism and statistical physics[1, 2, 3, 4, 5, 6, 7]. In geometrically frustrated antiferromagnets, spins usually sit on lattices made up of triangles as elementary units, and interact antiferromagnetically with their neighboring spins. Intrinsic inability to simultaneously satisfy all antiferromagnetic nearest-neighbor interactions on a triangle necessarily leads to macroscopic frustration. This makes the spin ordering on these lattices a highly nontrivial issue. Recent studies have revealed that frustration often gives rise to new interesting phenomena in the magnetic ordering, *e.g.*, phase transitions of novel universality classes, exotic ordered phases with novel order-der parameter, and the spin-liquid phase stabilized down to extremely low temperatures, *etc*. In this short review, I wish to deal with both the *classical* and *quantum* "chiral" phase transitions realized in certain frustrated XY-like antiferromagnets.

One interesting consequence of spin frustration in vector spin systems is the possible appearance of "chiral" degrees of freedom [3, 7, 8]. "Chirality" is a multispin quantity representing the sense or the handedness of the noncollinear spin structures induced by spin frustration. Two different types of chiralities have often been discussed in the literature: One is a *vector chirality* and the other a *scalar chirality*.

Chiral states representing the right- and left-handed configurations are illustrated in Fig.1 for an example of three antiferromagnetically coupled XY spins located at each corner of a triangle. The ground-state spin configuration is a well-known 120° spin structure, in which each XY spin on a plane makes an angle equal to ±120° with the neighboring spins. Here, one may define the chirality of the first type, the vector chirality, via a vector product of the two neighboring spins, averaged over three spin pairs, by

$$\kappa = \frac{2}{3\sqrt{3}} \sum_{<ij>} \left[\vec{S}_i \times \vec{S}_j \right]_z ,\tag{1}$$

where the summation is taken over three pairs of sites along the sides of the triangle in a clockwise direction. Evidently, the sign of κ represents each of the two chiral states, *i.e.*, either a right-handed (clockwise) state for $\kappa > 0$ or a left-handed (counterclockwise)

state for $\kappa < 0$. In the case of XY spins, the vector chirality κ is actually a *pseudoscalar* from a symmetry viewpoint: It remains invariant under global $SO(2) = U(1)$ proper spin rotations while it changes sign under global Z_2 spin reflections. Hence, in order to transform one chiral state to the other, one needs to make a mirroring operation, *i.e.*, a global spin reflection. The chiral order is then closely related to the spontaneous breaking of a discrete Z_2 spin-reflection symmetry.
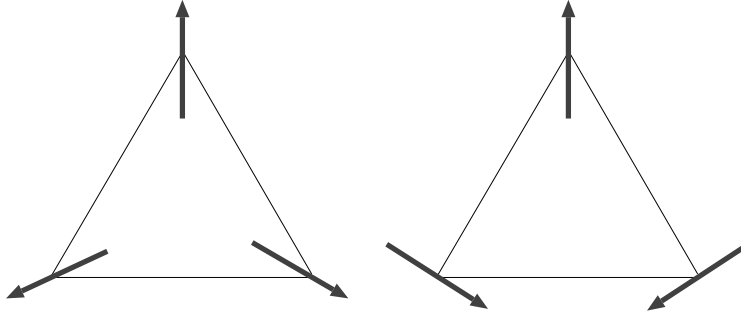

Figure 1: Two chiral states in the ground-state spin configurations of antiferromagnetically coupled three XY spins on a triangle. These two chiral states are characterized by the mutually opposite signs of the vector chirality.

In the case of three-component Heisenberg spins, the second type of chirality, the scalar chirality, has been discussed. It is defined for three neighboring Heisenberg spins by $\chi = \vec{S}_1 \cdot \vec{S}_2 \times \vec{S}_3$. This scalar chirality takes a nonzero value for noncoplanar spin states, its sign representing whether the noncoplanar structure is either right- or left-handed. Note that, in contrast to the vector chirality, the scalar chirality vanishes for any coplanar spin structure even when it is noncollinear. This scalar chirality is invoked in recent studies of magnetic properties of geometrically frustrated antiferromagnets [9] and spin glasses[7, 10, 11, 12, 13], but also of transposrt properties in manganites or pyrochlore magnets[14, 15, 16, 17, 18].

Our main concern in this article is the ordering of geometrically frustrated XY antiferromagnets. Hence, we consider in the following possible chiral order *associated with the vector chirality*. We also focus in this article on the chiral order in *regularly* frustrated systems without any quenched randomness. Historically, possible chiral order of frustrated XY antiferromagnets have been studied mainly for classical systems as a thermal ordering phenomenon. Thus, we first review in §2 the approaches performed for the antiferromagnetic XY models on various triangle-based lattices in one, two and three spatial dimensions, *i.e.*, the one-dimensional (1D) triangular-ladder lattice, the two-dimensional (2D) triangular lattice and the three-dimensional (3D) stacked-triangular lattice. All these lattices consist of triangles as elementary units, and the antiferromagnetic XY models defined on these lattices possess nontrivial chiral degrees of freedom as illustrated in Fig.1. As is well-known in theory of critical phenomena, the spatial dimensionality is crucially important in determining the nature of phase transition. Indeed, it has turned out that chiral order largely changes its nature depending on the spatial dimensionality of the lattice.

In the last decade, quantum phase transitions and quantum critical phenomena have attracted a lot of attention in various branches of condensed-matter and statis-

tical physics. Under such circumstances, it would be quite natural to ask what is the nature of the *quantum* chiral order, possibly realized in the ground state of purely quantum systems. In other words, it is possible to realize chiral order via a pure quantum phase transition with varying some parameters of the Hamiltonian at zero temperature? If yes, what is its nature in comparison with the thermal chiral order? Indeed, such studies of quantum chiral order was made extensively in these last few years. As an example of such recent studies, we wish to review in §3 theoretical studies on the chiral order of frustrated quantum spin chains. Finally in §4, we give brief summary and discussion, and conclude the review.

2. Chiral order in classical XY systems

In this section, we consider the thermal chiral order in purely classical systems. In order to clarify the important role of spatial dimensionality, we deal with the 1D, 2D and 3D triangular-lattice XY models in the following subsections §2.1-3, respectively.

2.1 One-dimensional triangular-ladder lattice

Let us begin with the 1D example. The model we consider is the classical two-component XY (plane rotator) model on the triangular ladder, a linear array of alternating upward and downward triangles. Each site has four nearest neighbors. The Hamiltonian is given by

$$\mathcal{H} = J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j, \tag{2}$$

where the interaction is assumed to be antiferromagnetic ($J > 0$) and work only between nearest-neighboring spins, while $\vec{S}_i = (S_i^x, S_i^y)$ is a two-component unit vector located at the i-th site on the triangular ladder. One may define the chirality at each upward triangle on the ladder by Eq.(1).

This 1D model can be solved exactly at arbitrary temperature by the standard technique, and the solution was reported by Horiguchi and Morita[19]. The ground state of this model is the 120° spin structure with either right-handed ($\kappa = 1$) or left-handed ($\kappa = -1$) chirality. Hence, at $T = 0$, the model exhibits a full long-range order (LRO) both in the spin and in the chiral sectors. Meawhile, since the model is a 1D one with short-range interaction, both the spin-spin and the chirality-chirality correlation functions remain short-ranged at any finite temperature without a finite-temperature transition of any type. Hence, the present model exhibits a zero-temperature phase transition both in the spin and in the chiral sectors.

The nontrivial issue is the manner how the spin and the chirality order at $T = 0$. Horiguchi and Morita observed by exact calculations that the spin-correlation length defined via the spin-spin correlation function $C_s(x) = < \vec{S}_0 \cdot \vec{S}_x > \approx A \exp(-x/\xi_s)$ (A being some constant) diverges with decreasing temperature as a power law, characterized by the associated spin-correlation-length exponent equal to unity, $\nu_s = 1$,

$$\xi_s \approx T^{-1}. \tag{3}$$

This divergence is common with the one observed in the unfrustrated 1D XY model. Meanwhile, the chiral-correlation length defined via the chirality-chirality correlation function $C_\kappa(x) = < \kappa_0 \kappa_x > \approx A' \exp(-x/\xi_\kappa)$ (A' being some constant) was found to diverge exponentially with temperature as

$$\xi_\kappa \approx \exp(-J/T), \tag{4}$$

which means the chiral-correlation-length exponent equal to infinity, $\nu_\kappa = \infty$. Such an exponential divergence happens to be common with the one observed in the standard 1D Ising model. A remarkable observation here is, though not necessarily be emphasized in Ref.[19], that the manners how the spin and the chirality correlations grow toward the $T = 0$ transition are mutually different, each characterized by mutually distinct correlation-length exponents, $\nu_s = 1$ *vs.* $\nu_\kappa = \infty$. This means that there exist two distinct diverging length scales in this $T = 0$ transition, one associated with the XY spin and the other associated with the chirality. The situation in in sharp contrast to that of the standard continuous (second-order) phase transitions characterized by only one diverging length scale (one-length-scaling hypothesis). Although the chirality is written as a product of two XY spins on short length scales of order lattice spacing, the chirality eventually outgrows the spin on long length scale, at least in an immediate vicinity of the $T = 0$ transition point, since $\nu_\kappa > \nu_s$ means $\xi_\kappa >> \xi_s$. We note that such an unusual situation, *i.e.*, the spin and the chirality exhibiting qualitatively different transition behaviors on long length scales, entails the "spin-chirality decoupling" or the "spin-chirality separation" on long length scale. Although both the spin and the chirality order simulataneously at $T = 0$ reflecting the 1D character of the model, the occurrence of the spin-chirality decoupling leads to an apparent violation of the one-length-scaling hypothesis, which, in turn, enables the spin and the chirality to exhibit mutually different transition behaviors.

In 2D, both the standard (unfrustrated) XY model and the standard Ising model are known to exhibit a finite-temperature transition. It would then be interesting to see whether the spin-chirality decoupling occurs in the frustrated XY model in 2D, and if it occurs, how the both order with decreasing temperature. Indeed, this problem has been studied quite extensively in the past fifteen years, which we will now review in the next subsection.

2.2 Two-dimensional triangular lattice

Typical example of the 2D frustrated XY model is the antiferromagnetic XY model on the triangular lattice [20, 21, 22, 23, 24, 25, 26]. We note that essentially similar physics is also expected to occur in some other models such as the fully-frustrated XY model on the square lattice [23, 27, 28, 29, 30, 31], or its dual counterpart (the Coulomb gas)[32, 33, 34], *etc.* While we present our discussion here in terms of the triangular-lattice XY antiferromagnet, the reader will find in cited references essentially similar results and controversy for these other models as well. The field-theoretical RG analysis was also applied to this 2D problem[35].

The spin and chirality ordering in the antiferromagnetic triangular-lattice XY model was first studied by means of Monte Carlo (MC) simulations by Miyashita and Shiba[20], and by Lee, Joannopoulos and Landau[21]. Miyashita and Shiba suggested that the spin and the chirality ordered at two close but separate finite temperatures. With decreasing temperature, the chirality ordered first at $T = T_\kappa$ characterized by the onset of the chiral LRO keeping the spin paramagnetic, and then at a slightly lower temperature $T = T_s < T_\kappa$, the spin exhibited a Kosterlitz-Thouless(KT) transition below which the quasi-LRO of XY spins developed and coexisted with the chiral LRO already established at a higher temperature $T = T_\kappa$. According to their scenario, the model exhibits a pure chiral phase at an intermediate temperature range $T_s < T < T_\kappa$ where the chirality exhibits a true LRO not accompanying the standard spin order. Obviously, such an ordering behavior requires the spin-chirality decoupling because the spin and the chirality order at different temperatures. According to Miyashita and Shiba, the criticality at the upper chiral transition at $T = T_\kappa$ was that of the standard 2D Ising model, with the

associated chiral exponents $\alpha = 0(\log)$, $\beta_\kappa = 1/8$, *etc*. Likewise, the criticality at the lower spin transition at $T = T_s$ was found to be of the standard KT universality, with the estimated spin-anomalous-dimension exponent $\eta = 0.25$ in agreement with the standard KT value.

In contrast, Lee *et al* suggested a somewhat different scenario for the same model[21]. According to these authors, the spin and the chirality ordered at a common finite temperature $T = T_c (= T_s = T_\kappa)$ where both the chiral LRO and the spin qiasi-LRO set in simultaneously. Yet, Lee *et al* suggested qualitatively different divergent behaviors to occur at $T = T_c$ for the spin-correlation length and for the chiral-correlation-length, the former exhibiting a power-law divergence with an exponent $\nu_\kappa \sim 1$ and the latter exhibiting the KT-like exponential divergence. This means that, in spite of the simultaneous occurrence of the spin and the chirality transitions, the model still exhibits the spin-chirality decoupling in the sense that there exist two distinct deverging length scales at the transition.

Numerous numerical works have been performed since then on the same and related models with the aim at clarifying the nature of the transition. While the controversy have continued, and this controversy has not yet been settled completely, recent numerical works tend to converge in that the spin and the chirality order at two close but separate temperatures, the chiral ordering preceding the spin ordering, $T_\kappa > T_s$ [24, 25, 26, 29, 30, 31, 33, 34]. The estimated T_κ and T_s are mutually close, the difference being only of order $(T_\kappa - T_s)/T_\kappa = 0.3\% \sim 3\%$, depending on the particular model and the authors. As such, it appears that frustrated 2D XY models generically possess a pure chiral phase in a narrow but finite temperature range. (It might also be worth mentioning that there exists a certain 2D model, a coupled Ising-XY-Heisenberg model invented to mimic the superfluidity transition of helium-three film, where the Z_2 and $U(1)$ orderings were observed to occur at widely separate temperatures, more than 8% apart [36].)

The issue of criticality of the spin and of the chirality, by contrast, remains more ambiguous. Some authors claim that the criticalities of the chirality and of the spin are of the standard Ising and KT ones[24, 25, 30], respectively, while others claim that they are distinct from the standard Ising and KT ones even when both order at two distinct temperatures[26, 29, 31, 34].

2.3 Three-dimensional stacked-triangular lattice

In this subsection, we wish to briefly touch upon the spin and the chirality orderings of the 3D triangular-lattice XY antiferromagnet. There are several ways to construct a 3D lattice by stacking the 2D triangular layers. We consider here the simplest construction which preserves the chiral $Z_2 \times U(1)$ symmetry, *i.e.*, the 3D stacked-triangular lattice (or a simple-hexagonal lattice) in which the 2D triangular layers are stacked in register on top of each other. Since there is no frustration along the orthogonal direction in this type of stacked-triangular lattice irrespective of the sign of the interplane interaction, the ordered-state spin configuration is a three-sublattice 120° spin structure in each triangular layer.

In 3D, we have several experimental realizations of the model at issue. Indeed, various stacked-triangular antiferromagnets with nontrivial chiral degree of freedom have been known: Examples are ABX_3-type compounds $CsMnBr_3$ and $CsVBr_3$. Even Ising-like ABX_3-type compounds with an easy-axis-type anisotropy, such as $CsNiCl_3$, $CsNiBr_3$ and $CsMnI_3$, exhibit the chiral critical behavior under external fields higher than a certain critical field. Extensive experimental measurements have been performed on these chiral XY-like antiferromagnets which have been summarized in several review articles[1, 2, 3, 7]. Recent experimental progress has made it possible even to directly

observe the chirality by using the polarized neutron-scattering technique[37, 38].

In sharp contrast to the 1D and 2D cases, there are good numerical and experimental evidence in 3D that the spin and the chirality ordered simultaneously in 3D via a single phase transition accompanied with the onset of the noncollinear spin LRO (120° structure). In particular, Plakhty observed by means of polarized neutron-scattering measurements on the triangular-lattice XY antiferromagnet $CsMnBr_3$ that the spin and the chirality indeed ordered simultaneously[38]. The next question would then be whether the simultaneous spin and chirality transition accompanies the spin-chirality decoupling or not, namely, whether $\nu_\kappa = \nu_s$ or $\nu_\kappa \neq \nu_s$ at the transition. In fact, the values of ν_s and ν_κ, estimated either numerically[39]or experimentally[38], turned out to be close to each other, suggesting that the equality $\nu_\kappa = \nu_s$ is likely to hold. Hence, in the case of the 3D stacked-triangular XY angiferromagnet, there occurs a single phase transition with a common spin- and chirality-correlation-length exponent $\nu = \nu_\kappa = \nu_s$ without the spin-chirality decoupling. The situation here is in sharp contrast to the 1D and 2D cases where the spin-chirality decoupling takes place. Such a difference may be understandable if one notes the following: Since the spin-chirality decoupling does not arise in the mean-field limit corresponding to an infinite dimension, higher dimensionality generally tends to suppress the spin-chirality decoupling and to recover the conventional transition behavior with a common diverging length scale occurring in both the spin and in the chiral sectors. This suggests that strong fluctuations borne by the combined effects of low dimensionality and frustration should be crucial in realizing the spin-chirality decoupling.

Even if the 3D chiral transition is conventional in the above sense, we note that the associated criticality may well be non-standard. Rather, the chiral $Z_2 \times U(1)$ symmetry simultaneously broken at the transion might give rise to the non-standard criticality, possibly described by a new type of fixed point (chiral fixed point). Indeed, this was the proposal made some time ago by the present author: On the basis of a symmetry argument [39, 40], Monte Carlo simulations [39, 41] and renormalization-group (RG) calculations[42], possible occurrence of such new chiral universality class was suggested for the 3D chiral XY system. While various experiments[2, 3, 7, 43, 44, 45, 46, 47, 48, 49, 50, 51] performed on the 3D stacked-triangular XY antiferromagnet without lattice distortion (e.g., $CsMnBr_3$) generally support this conjecture, several theoretical works claimed that the transition should in fact be weakly first order, and the situation remains controversial. We donot intend here to enter into further details of the controversy, nor to give a complete list of references. The reader is invited to several recent review articles.[1, 3, 7] (Some of the very recent theoretical works have not been included in these reviews: Mentioning only a few of them, six-loop RG calculation favors the chiral-universality scenario[52], while the so-called "exact RG" calculation favors the weak first-order transition[53], etc.)

I wish to conclude this section by summarizing the ordering properties of the classical chiral XY systems presented in each subsection. In 1D and 2D, the spin-chirality decoupling takes place. The spin and the chirality show mutually different transition behaviors. This is in contrast to the 3D case where the spin-chirality decoupling does not occur. In 1D, both the spin and the chirality order simultaneously at $T = 0$, but with mutually different correlation-length exponents, $\nu_\kappa > \nu_s$. In 3D, both the spin and the chirality order simultaneously at a finite temperature. Unlike the 1D case, there is only one diverging length scale at this transition, the spin and the chirality possessing a common correlation-length exponent $\nu_\kappa = \nu_s$. In 2D, recent works strongly suggest that the spin and the chirality order at two close but separate temperatures, $T_\kappa > T_s$.

This means that in 2D there occurs a pure chiral phase at an intermediate temperature regime, $T_s < T < T_\kappa$, where only the chirality exhibits a LRO keeping the XY-spin paramagnetic.

3. Chiral order in quantum XY systems

It sometimes happens that the D-dimensional quantum system at zero temperature can be mapped onto the $D + 1$-dimensional classical system at finite temperature. For example, thermodynamic properties of certain $D + 1$-dimensional classical system at finite temperature embodied in the maximum eigenvalue of the associated transfer matrix can often be mapped onto the ground-state properties of appropriate D-dimensional quantum system. Of course, in order to substantiate the correspondence, such an analogy has to be examined carefully in each particular case. Nevertheless, one may make a first guess that various thermal chiral order identified in classical systems might have some counterparts in the corresponding quantum systems whose spatial dimension is one-dimension less than the classical ones. In view of the property of the 2D classical XY system reviewed in §2.2, one may then imagine that the 1D frustrated quantum XY spin chain might exhibit quantum chiral order at $T = 0$ with varying a suitably defined parameter of the Hamiltonian. Motivated by such an expectation, we recently undertook a systematic numerical investigation of the frustrated quantum XY spin chain based on the exact-diagonalization and the density-matrix-renormalization-group (DMRG) methods[54, 55, 56, 57]. We have then found that the above naive expectation based on the classical-quantum analogy does indeed hold.

Since the details of the calculations have already been given in Refs.[54, 55, 56] and in a recent review article[57], we sketch here only the gross features of the ordering properties of the model. The Hamiltonian considered is

$$\mathcal{H} = \sum_{\rho=1,2} J_\rho \sum_l \left(S_l^x S_{l+\rho}^x + S_l^y S_{l+\rho}^y \right), \tag{5}$$

where $J_1 > 0$ and $J_2 > 0$ are the antiferromagnetic nearest-neighbor and next-nearest-neighbor couplings along the 1D chain, while $\vec{S}_i$ is now the spin-S quantum-mechanical operator. In the special case of $J_1 = J_2$, the model can be regarded as the triangular-ladder model with the nearest-neighbor antiferromagnetic coupling as considered in §2.1. Here, we extend the triangular-ladder model, or the $J_1 = J_2$ model on a single chain, by introducing the independent nearest- and next-nearest-neighbor interactions on a single chain. This enables us to have a free parameter $j \equiv J_2/J_1$ in the Hamiltonian, which controls the extent of frustration, and eventually, drives the chiral order in the ground state. Remember we focus in this section on the ground-state properties of the model, since we are interested in pure quantum phase transition. The local chirality may be defined as a quantum-mechanical operator defined by

$$\kappa_i = S_i^x S_{i+1}^y - S_i^y S_{i+1}^x. \tag{6}$$

We note that the possible chiral order of this model has also been studied analytically on the basis of the field-theoretical method by Nersesyan et al[59], by Lecheminant et al[60], and by Kolezhuk[61].

Below, we summarize the properties revealed mainly by the exact-diagonalization and DMRG methods [54, 55, 56].

i) The case of $S = 1/2$: With increasing j from $j = 0$ to $j = \infty$, there appear three distinct phases, $i.e.$, the spin-fluid phase, the dimer phase and the gapless chiral phase.

The gapless chiral phase possesses a true chiral LRO with algebraically-decaying spin correlations. In the numerical accuracy of Ref.[56], the possible gapped chiral phase (or the chiral dimer phase) was not identified. The two transitions associated with the spin fluid-dimer transition and with the dimer-gapless chiral transition are both continuous. Even for higher half-integer $S \geq 3/2$, qualitative features of the phase structure remain the same.

ii) The case of $S = 1$: With increasing j from $j = 0$ to $j = \infty$, there appear three distinct phases, $i.e.$, the Haldane phase, the gapped chiral phase (or the chiral Haldane phase) and the gapless chiral phase. The latter two phases are chiral ordered phases possessing a true chiral LRO. The gapped chiral phase has exponentially-decaying spin correlations, while the gapless chiral phase has algebraically-decaying spin correlations. The two transitions associated with the Haldane-gapped chiral transition and with the gapped chiral-gapless chiral transition are both continuous. We note that the phase structure observed here is quite similar to the one observed in the classical 2D frustrated XY model as reviewed in §2.2 with varying temperature: Large-, intermediate- and small-j phases of the quantum 1D model correspond to low-, intermediate- and high-temperature phases of the classical 2D model. For higher integer $S \geq 2$, there appears in addition the spin-fluid phase for smaller values of j.

We note that the results of field-theoretical analyses and numerical calculations made so far are consistent with each other on most of the above points. One ambiguity still being left might be whether there exists a gapped chiral phase (chiral dimer phase) for half-odd-integer S. Field theory claims that there should exist such a phase in a narrow interval of j between the dimer and the gapless chiral phases[60], while the numerical DMRG calculation could not identify such a phase within the numerical accuracy[56]. This points needs further clarification. Quantum chiral order was now studied for more general types of anisotropy, $e.g.$, an XXZ-type ansiotropy[55, 56] and a single-ion-type anisotropy[58]. Anyway, owing to the recent analytical and numerical studies, the existence of quantum chiral order in the frustrated 1D quantum XY-spin chain has now been well established. We note in passing that a similar 1D quantum-2D classical analogy was also examined in terms of a Josephson-junction array in a magnetic field[62, 63]. In this case, the charging effect of superconducting grains plays the role of the quantum effect.

4. Concluding remark

A brief review has been given on the recent works on chiral order in regularly frustrated XY systems both in classical and quantum cases. In the classical case, chiral transitions of the 1D, 2D and 3D triangular-lattice antiferromagnets have been examined. In 1D and 2D, the spin-chirality decoupling phenomenon takes place at the transition, while in 3D the chiral transition satisfies the standard one-length-scaling hypothesis without the spin-chirality decoupling phenomenon, though the associated fixed point might well be novel because of the underlying chiral symmetry (a chiral fixed point). In 2D, there appears a pure chiral phase in an intermediate range of temperature where the chirality exhibits a LRO with keeping the XY spin disordered. In the quantum case, the $T = 0$ chiral transition of the 1D spin-S XY model ($J_1 - J_2$ or zigzag chain) has been examined. There exist two types of chiral phases, gapped and gapless chiral phases, where the chirality exhibits a LRO. The gapless chiral phase with algebraically-decaying spin correlations exists for general S, while the gapped chiral phase with exponentially-decaying spin correlations is identified only for integer-S. Analogy between the quantum chiral order in D-dimensions and the thermal chiral order in $D + 1$-dimensions is discussed.

In view of the 1D-quantum and 2D-classical analogy, it would be natural to extend it to the systems one-dimension higher, *i.e.*, to the 2D-quantum and 3D-classical analogy. Then, one expects that the frustrated 2D quantum system may well exhibit a magnetic phase transition of chiral universality class. In fact, such a possibility has already been examined in several 2D quantum phase transitions, including those of Josephson-junction array in a magnetic field[64], frustrated Heisenberg antiferromagnet[65], square-lattice bilayer Heisenberg model[66], and triangular-lattice bilayer Heisenberg model[67].

Although we have confined ourselves in this review to the chiral order associated with the vector chirality in XY-like systems, there also exist several examples of Heisenberg-like systems where the chiral order associated with the scalar chirality takes place. We have also confined ourselves to the regularly frustrated systems here, neglecting the effects of quenched randomness. There are several occasions, however, where the quenched randomness plays an essential role in the chiral ordering. Examples are the ordering of vector spin glasses, including both Heisenberg[10, 11, 12, 13] and XY[68, 69] spin glasses, and of ceramic high-T_c superconductors[70, 71]. In fact, quenched randomness generally tends to enhance fluctuations and serves preferably to cause the spin-chirality decoupling and to realize the pure chiral phase (chiral-glass phase). The related works have been briefly reviewed in Ref.[7].

Thus, the chiral order, both thermal and quantum, is likely to be realized in a rather wide class of frustrated systems, giving rise to intriguing ordering behaviors. Much needs to be done in the future to fully explore this rich field.

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